Available on leon2k2k2k@github.io.

$\mathbb{E}_n$ algebras in $(m+1)$-Categories

Forthcoming joint work with Amartya Dubey

Outline: §1: Motivation & main results.
§2: Sketch of proof

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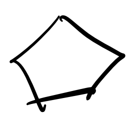
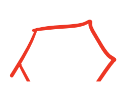
§1.1: Motivation: how to construct $\mathbb{E}_n$ algebras in $(m+1)$-categories.

$\mathcal{C}$ is a $(m+1)$-category if $\forall\, c,d \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(c,d)$ is m-truncated.

e.g.

	2-category
$\mathbb{E}_1$	$\otimes$, α,  + units.
$\mathbb{E}$	β   $\times 2$: 

$\mathbb{E}_2$	[illegible]
$\mathbb{E}_3 = \mathbb{E}_\infty$	$\beta^2 = id.$

Q: how to generalize this to general m-categories?

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§1.2: $\mathbb{E}_1$-algebras in m-categories?

We already have answer:

[Stasheff 63]

$$\mathbb{E}_0 = \mathbb{A}_1 \to \mathbb{A}_2 \to \mathbb{A}_3 \to \mathbb{A}_4 \cdots \to \mathbb{A}_\infty = \mathbb{E}_1$$

A. Going from $\mathbb{A}_n \to \mathbb{A}_{n+1}$, we first fill in a $K_n \simeq S^{n-2} \to D^{n-1}$ cell in the $\mathrm{Hom}(C^{\otimes n+1}, C)$

non-unital associahedron

then higher cells for units.

B. $\mathbb{A}_n \to \mathbb{A}_{n+1}$ is $(n-3)$-connected, and

is equivalence on $\mathbb{A}_n(k) \to \mathbb{A}_{n+1}(k)$ for is equivalence on $\mathbb{A}_n(k) \to \mathbb{A}_{n+1}(k)$, for $k \le n$.

Very important

$f: \mathcal{P} \to \mathcal{Q}$ map of operads is

① n-connected if $\forall X_1, \dots X_n; Y \in \mathcal{P}$, $\mathrm{Mul}_{\mathcal{P}}(X_1, \dots X_n; Y) \to \mathrm{Mul}_{\mathcal{Q}}(fX_1, \dots fX_n; fY)$ is n-connected.

② n-truncated if $\forall X_1, \dots X_n; Y \in \mathcal{P}$, $\mathrm{Mul}_{\mathcal{P}}(X_1, \dots X_n; Y) \to \mathrm{Mul}_{\mathcal{Q}}(fX_1, \dots fX_n; fY)$ is n-truncated

③ $\mathcal{O}$ is $(m+1)$-operad if $\mathcal{O} \to \mathbb{E}_\infty$ is m-truncated.

④ $\mathcal{C}$ sym. mon. ∞-cat, then $\mathcal{C}^{\otimes}$ is $(m+1)$-operad iff $\mathcal{C}$ is a $(m+1)$-category.

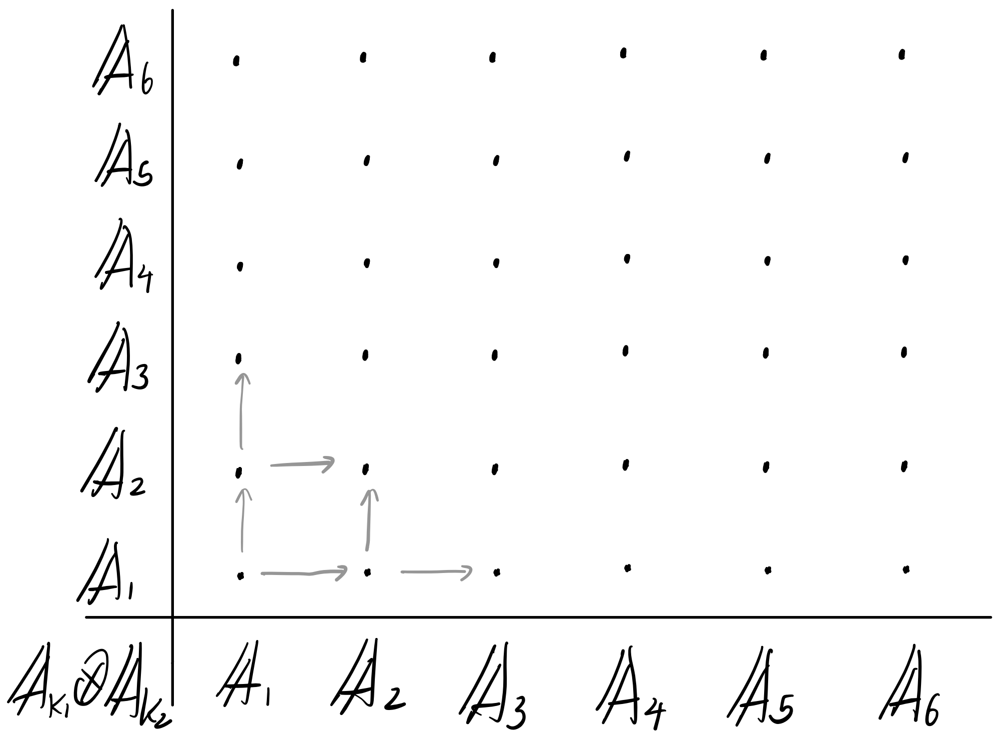
⑤ $\mathcal{P} \to \mathcal{Q}$ m-truncated, $\mathcal{C}$ sym. mon. $(m+1)$-cat $\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{P}}(\mathcal{C})$ is an equivalence.

This implies that

Cor: $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{A}_{m+1}}(\mathcal{C})$, when $\mathcal{C}$ is a $(m+1)$-category.

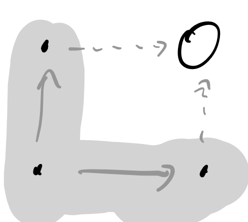
§1.3. $\mathbb{E}_2$-algebras in $(m+1)$-categories.

Dunn additivity: $\mathbb{E}_2 \simeq \mathbb{E}_1 \otimes \mathbb{E}_1$, therefore it has a filtration by $\mathbb{A}_{k_1} \otimes \mathbb{A}_{k_2}$:



Getting $\mathbb{E}_2$ = filling in each cell.

Note: we fill cell relative to its left and bottom:



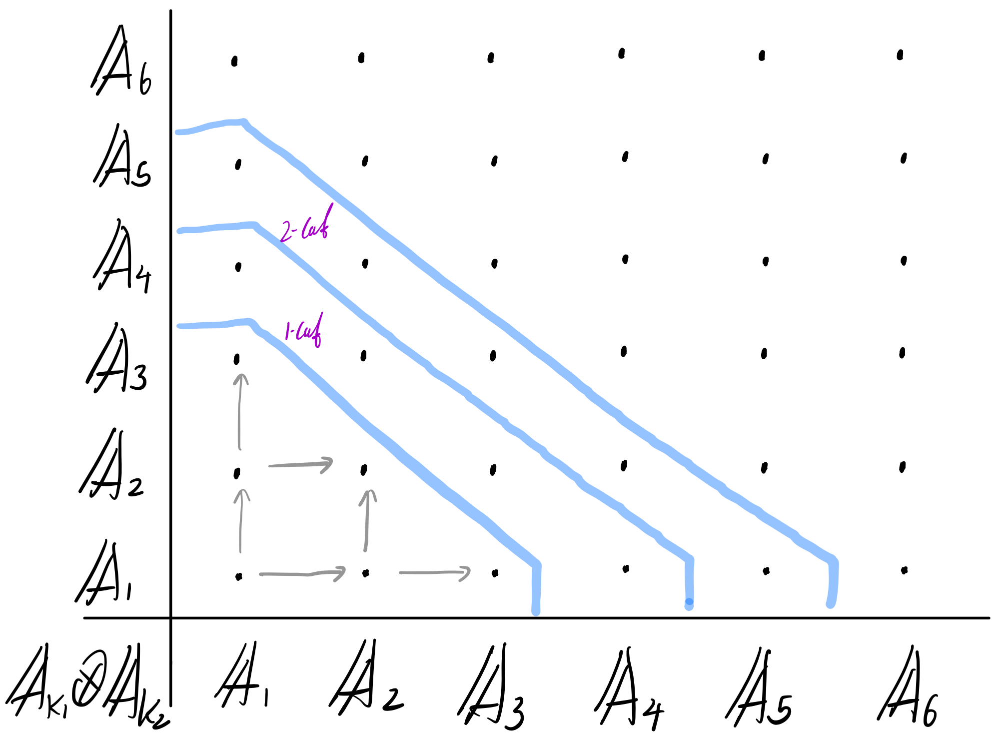
$$\begin{tikzcd}
\mathbb{A}_{k_1}\otimes\mathbb{A}_{k_2+1} \arrow[r, dashed] & \mathbb{A}_{k_1+1}\otimes\mathbb{A}_{k_2+1} \\
\mathbb{A}_{k_1}\otimes\mathbb{A}_{k_2} \arrow[u] \arrow[r] & \mathbb{A}_{k_1+1}\otimes\mathbb{A}_{k} \arrow[u, dashed]
\end{tikzcd}$$

Naive attempt: Cell counting

We are filling in $S^{k_1-2}\times D^{k_2-1} \dashrightarrow D^{k_1-1}\times D^{k_2-1}$

$$S^{k_1-2}\times S^{k_2-2} \longrightarrow D^{k_1-1}\times S^{k_2-2},$$

aka $S^{k_1+k_2-3} \to D^{k_1+k_2-1}$. Therefore each cell is (k_1+k_2-4)-connected. This means we need to fill in triangles.



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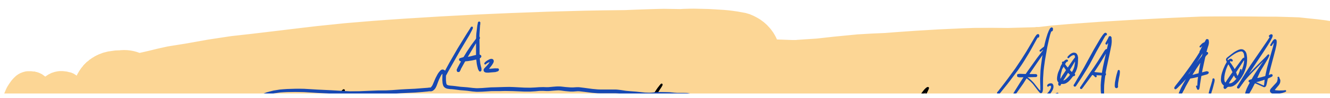
§1.4 Eckmann-Hilton arguments:

Let's recover the classical EH argument:



Given 2 (unital) binary operation on cGSet, $(C,\otimes,1_\otimes)$, $(C,*,1_*)$, such that $\forall x,y,z,w$:

$$(x\otimes y)*(z\otimes w) = (x*z)\otimes(y*w),$$

$A_2\otimes A_2$

then

① $\otimes = *$, $1_\otimes = 1_*$

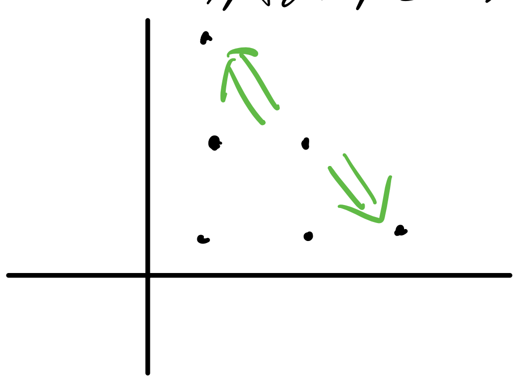
② $*$ (thus $\otimes$) is associative ($A_3\otimes A_1$)

③ $*$ is commutative ($\mathbb{E}_2$).

$y=1$

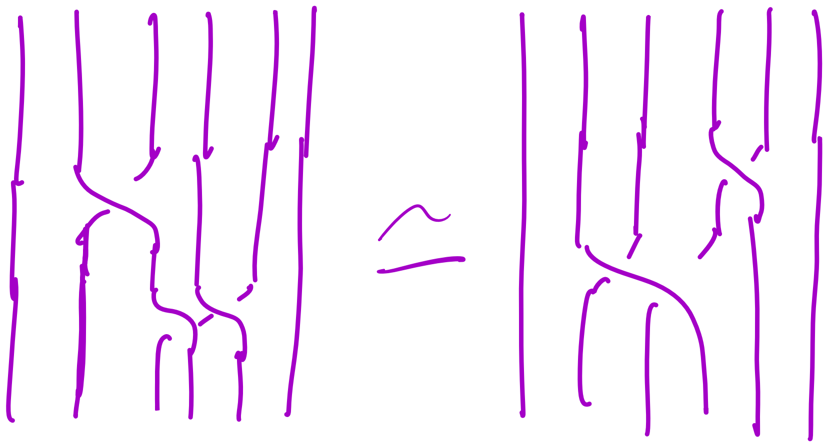
$$(x*1)*(z*w) = (x*z)*(1*w).$$

We see $A_2\otimes A_2 \Rightarrow A_3\otimes A_1$ and $A_1\otimes A_3$

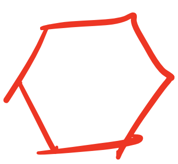
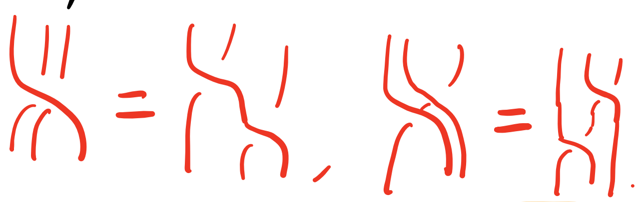


by plugging in units.

Let's do another example, let's consider $A_2\otimes A_2$, which ask the following:

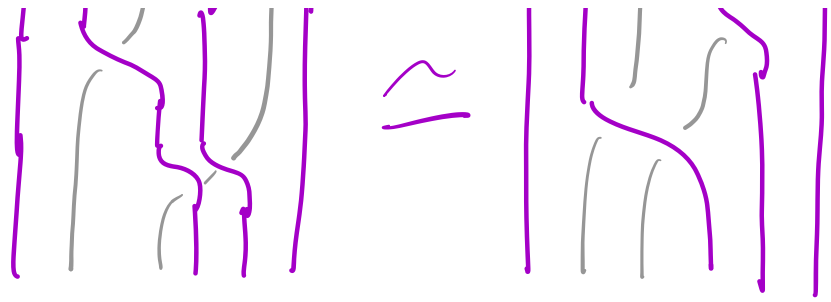


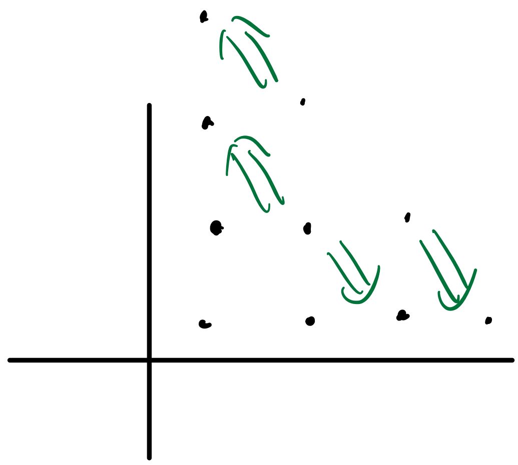
Note this is equivalent to both of the

 ×2 : 

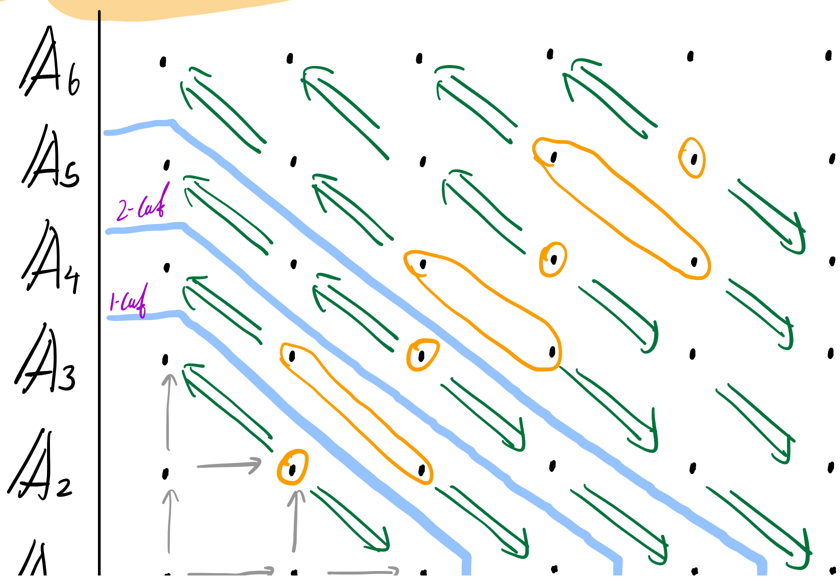
Remark: Braided monoidal is really equivalent to $A_3 \otimes A_2$.

Exercise: $A_3 \otimes A_2 \Rightarrow A_4 \otimes A_1$. Pentagon by





Eckmann-Hilton I: On diagonal implies off-diagonal.



As a Consequence:

$$\underline{\mathbb{H}}_2 = (\,?\,)$$

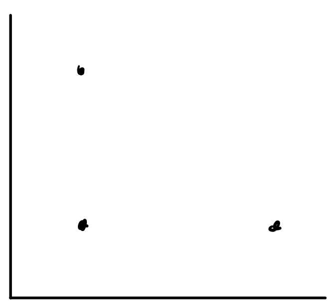
1-Cat: $\mathbb{A}_2 \otimes \mathbb{A}_2$

3-Cat: $\mathbb{A}_3 \otimes \mathbb{A}_3$.

$\vdots$

But what about the even case?

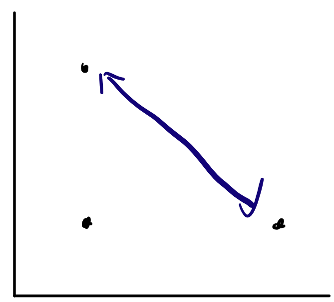
Baby Case:



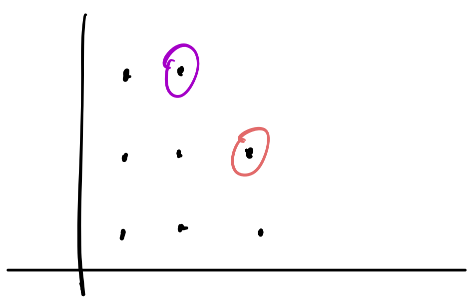
This ask for $(\mathcal{C}, \otimes \xrightarrow{1} \mathcal{C})$ and 2 unital binary structure on it. Note that given $(\mathcal{C}, \otimes)$ we can just make the other operation also $(\mathcal{C}, \otimes)$.

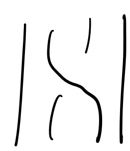
In fact, by Eckmann-Hilton, the 2 operations has to agree anyways.

We represent this as

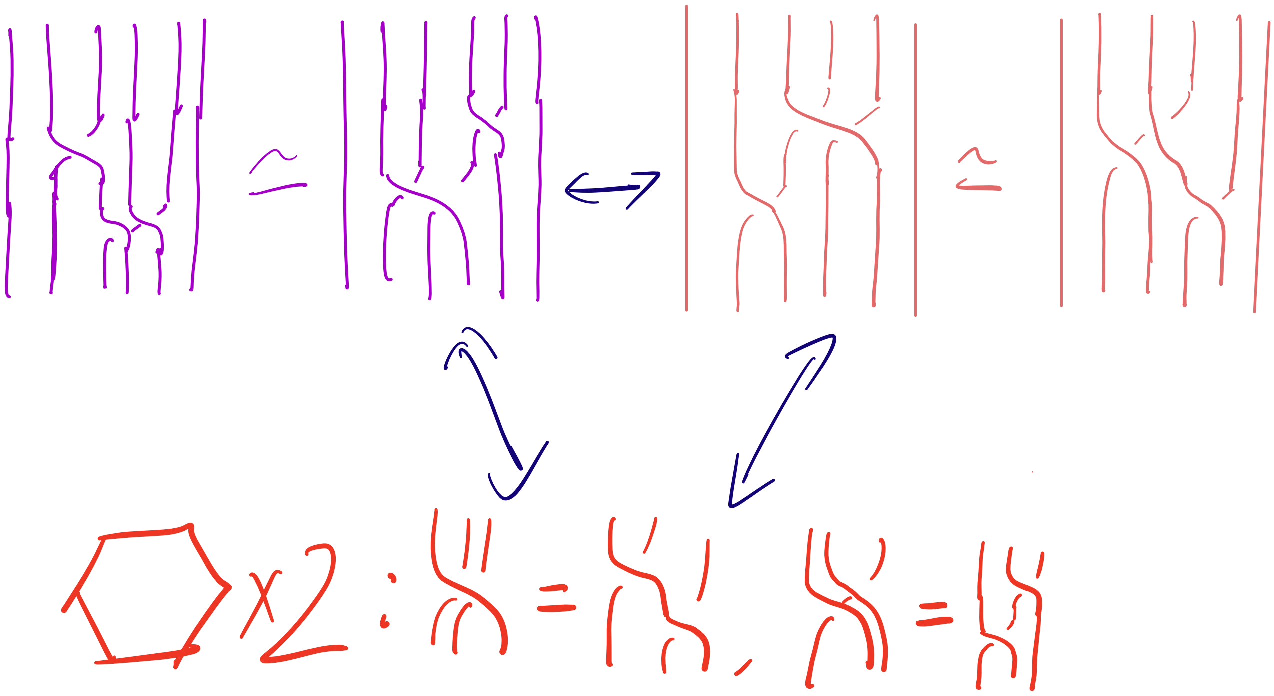


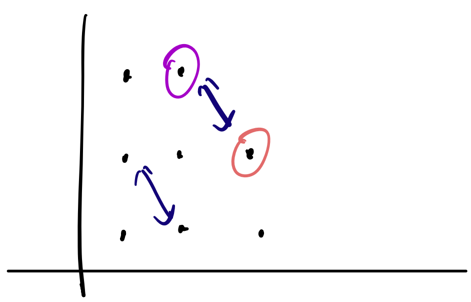
Well for 2-cat, it seems like we need



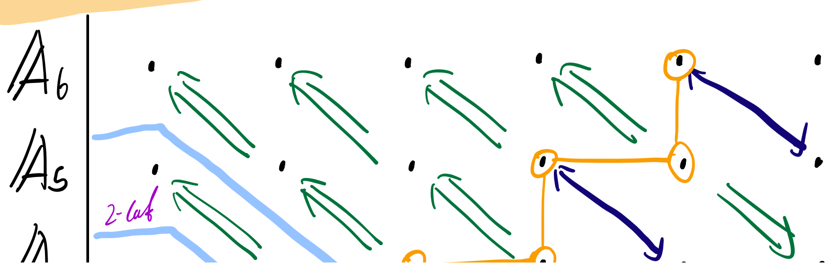
However, $A_2 \otimes A_2$ is braiding  , and

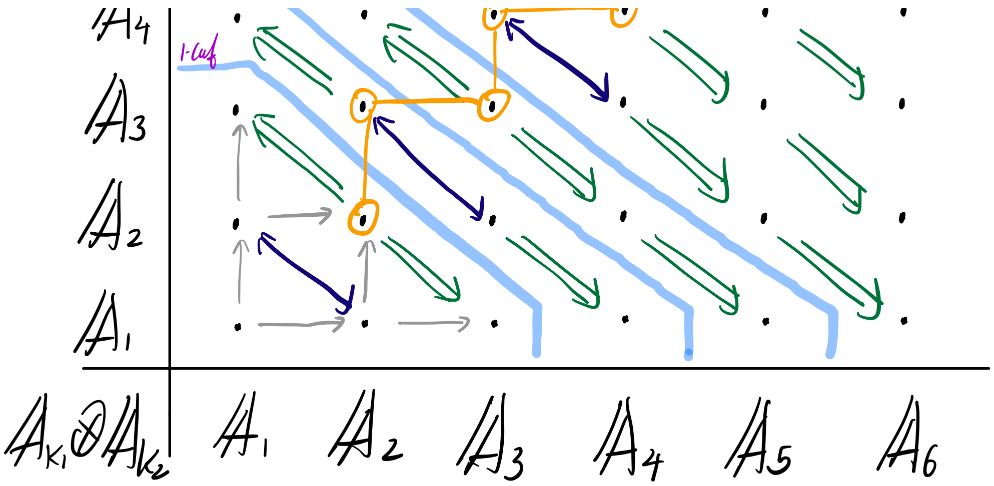
$A_1 \otimes A_2$ $A_2 \otimes A_3$





Eckmann-Hilton 2: there is a reflection sym. around the diagonal





Therefore it is suffice to go up the stairs

$\mathbb{E}_2$	$\mathbb{A}_{k_1}\otimes\mathbb{A}_{k_2}$	
1-cat	$(2,2)$	
2-cat	$(3,2)$	braided monoidal
3-cat	$(3,3)$	
4-cat	$(4,3)$	

Slogan: $\mathbb{E}_2$-alg in m-cat: go up while hugging the diagonal "as close as possible"

This generalizes to $\mathbb{H}_k$, where we also want

$\mathbb{A}_{n_1} \otimes \cdots \otimes \mathbb{A}_{n_k} \to \mathbb{H}_k$ with $(n_1, \dots n_k)$ as close to diagonal as possible:

$\mathbb{H}_3$	$\mathbb{A}_{k_1} \otimes \mathbb{A}_{k_2} \otimes \mathbb{A}_3$
1-cat	$(2,2,1)$
2-cat	$(2,2,2)$
3-cat	$(3,2,2)$
4-cat	$(3,3,2)$
5-cat	$(3,3,3)$.

Here's the result:

Thm (Y.L., Dubrey): given $n_1 \le n_2 \le n_3 \le \cdots \le n_k$, with first i equal.

$\mathbb{A}_{n_1} \otimes \mathbb{A}_{n_2} \cdots \otimes \mathbb{A}_{n_k} \to \mathbb{E}_k$ is $(kn_1 - 2 - i)$-connected*.

§2: Sketch of proof

$k=2$. The result comes from inductive consider

$$\mathbb{A}_{k_1} \otimes \mathbb{A}_{k_2} \to \mathbb{A}_{k_1+1} \otimes \mathbb{A}_{k_2}$$

It is of the form $P \otimes R \to Q \otimes R$.

There are 3 things:

1. R connectivity (and coherentness)
2. $P \to Q$ connectivity.
3. $P_{\leq k} \to Q_{\leq k}$ is an equivalence

See [SY19].

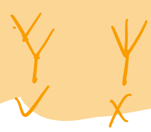
It's the interplay of these three that makes result work. Let's quickly review part 3:

§2.1: k-restricted operads.

A k-restricted operad $\mathcal{O}_{\le k}$ can be defined 2 ways

① (Lurie) $\mathcal{O}_{\le k} \to \mathrm{Fin}_{*,\le k}$ with ...

② (Dendroidal) A Segal presheaf on $\Omega_{\le k}$ ← trees with valence $\le k$.



for $k=2$

There is a restriction

$$\mathrm{Op} \underset{(-)_{\le k}}{\overset{L_k}{\rightleftarrows}} \mathrm{Op}_{\le k}$$

with left adjoint L_k:

$L_k \mathcal{O}_{\le k}(n) = \{$ all ways to create n-ary from

Σk-ary morphism$\}$.

However, this is also a right adjoint, when restricted to unital operad.

$$\begin{tikzcd} \mathrm{Op}^{un} \arrow[r, bend right, "R_k"'] \arrow[r, bend left, <-, "L_k"] & \mathrm{Op}^{un}_{\Sigma k} \arrow[l, <-] \end{tikzcd}$$

Idea: $R_k \mathcal{O}_{\Sigma k}(n) = \left\{ \begin{array}{l} \text{all } k\text{-ary info a } n\text{-ary morphism} \\ \text{contains} \end{array} \right\}$

$= \left\{ \begin{array}{l} \text{all ways to get } k\text{-ary mor by} \\ \text{plugging in units} \end{array} \right\}$.

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e.g. $k=1$. $\mathrm{Op}^{un}_{\Sigma k} \overset{\text{forget units}}{\simeq} \mathrm{Op}^{nu}_{\Sigma_1} \simeq \mathrm{Cat}_\infty$.

the right adjoint

$C^{\amalg}\ \ R_k C(X_1, \dots, X_n; Y) = \mathrm{Hom}(X_1, Y) \times \dots \times \mathrm{Hom}(X_n, Y)$

$C = R_1 C(X_1, \dots X_n; Y) = \mathrm{Hom}_C(X_1; Y) \times \cdots \times \mathrm{Hom}_C(X_n; Y)$

plug units in all but 1 spot.

In general:

Prop: C sym. mon. unital ∞-cat with colimits:

$$R_k C^{\otimes}_{\leq k}(X_1, \dots X_n; Y) = \mathrm{Hom}_C(\operatorname{colim} F_{X_1, \dots, X_n}|_{\leq k}; Y).$$

where $F_{X_1, \dots X_n} : P(\{1, \dots, n\}) \to C$ is

$$\begin{tikzcd}
 & X_1 \times X_2 \arrow[rr] & & X_1 \times X_2 \times X_3 \\
X_1 \arrow[ur] \arrow[rr] & & X_1 \times X_3 \arrow[ur] & \\
 & X_2 \arrow[uu] \arrow[rr] & & X_2 \times X_3 \arrow[uu] \\
* \arrow[uu] \arrow[ur] \arrow[rr] & & X_3 \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

$|_{\leq k}$ means restrict to $J \subset \{1, \dots n\}$ w/ $|J| \leq k+1$.

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§ 2.2 Proof sketch

§2.2: proof sketch

Thm: $f: P \to Q$ d_1-conn, $P_{\le k} \to Q_{\le k}$ equivalence, and R d_2-connected*, then

$$P \otimes R \to Q \otimes R$$

is $(d_1 + k(d_2 - 2))$-connected.

* means coherent.

Pf: $D = d_1 + k(d_2 - 2)$. Suffice $C = S_{\le D}$.

$$\begin{tikzcd} P \otimes R \arrow[r] \arrow[d] & C^{\otimes} \arrow[d] \\ Q \otimes R \arrow[r] \arrow[ur, dashed] & E_\infty \end{tikzcd} \iff \begin{tikzcd} P \arrow[r] \arrow[d] & \mathrm{Alg}_R(C^{\otimes}) \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & E_\infty \end{tikzcd}$$

$P_{\le k} = Q_{\le k}$,

$$\begin{tikzcd} P \arrow[r] \arrow[d, "d_1\text{-conn.}"] & \mathrm{Alg}_R(C^{\otimes}) \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & R_K(\mathrm{Alg}_R(C^{\otimes})_{\le k}) \end{tikzcd}$$

Suffice this is d_1-trun.

Looking at space of lifts

$$\begin{tikzcd}
\operatorname{colim} \mathcal{F}_{X_1 \cdots X_n}|_{\leq k} \arrow[r] \arrow[d, "?\text{-conn}"'] & Y \arrow[d, "D\text{-truncated}"] \\
X_1 \times \cdots \times X_n \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

The space of lifts is $(D-?+2)$-truncated, so want

$? = k(d_2+2)-2.$

So suffice to show

$$\operatorname{colim} \mathcal{F}_{X_1 \cdots X_n}|_{\leq k} \to X_1 \times \cdots X_n \text{ is}$$

$(k(d_2+2)-2)$-connected.

$$\begin{tikzcd}
& X_1 \times X_2 \arrow[rr] & & X_1 \times X_2 \times X_3 \\
X_1 \arrow[ur] \arrow[rr] & & X_1 \times X_3 \arrow[ur] & \\
& X_2 \arrow[uu] \arrow[rr] & & X_2 \times X_3 \arrow[uu] \\
* \arrow[uu] \arrow[ur] \arrow[rr] & & X_3 \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Note: strongly Cartesian, so a Blakers-Massey result.

Baby case: $X_1 \amalg X_2 \to X \times Y$ is (-1)-conn for $\mathbb{E}_1$-alg.

Prop (Y.L, Dubey): R d-conn. coherent, $X_1 \cdots X_n \in \mathrm{Alg}_R(S)$

$\operatorname{colim}\{X_1 \cdots X_n\}_{\le k} \to X_1 \times \cdots \times X_n$ is

$(k(d_2+2)-2)$-connected.